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# Canonical quantization of a particle near a black hole *

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## Abstract

We discuss the quantization of a particle near an extreme Reissner-Nordström black hole in the canonical formalism. This model appears to be described by a Hamiltonian with no well-defined ground state. This problem can be circumvented by a redefinition of the Hamiltonian due to de Alfaro, Fubini and Furlan (DFF). We show that the Hamiltonian with no ground state corresponds to a gauge in which there is an obstruction at the boundary of spacetime requiring a modification of the quantization rules. The redefinition of the Hamiltonian *à la* DFF corresponds to a different choice of gauge. The latter is a good gauge leading to standard quantization rules. Thus, the DFF trick is a consequence of a standard gauge-fixing procedure in the case of black hole scattering.

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## I. INTRODUCTION

The simplest quantum mechanical system with conformal symmetry is described by the Hamiltonian

$$H = \frac{p^2}{2} + \frac{g}{2x^2} \tag{1}$$

It is easy to see that this Hamiltonian possesses a continuous spectrum down to zero energy and there is no well-defined ground state. A solution to this problem was suggested by de Alfaro, Fubini and Furlan (DFF) a long time ago [1]. They proposed the redefinition of the Hamiltonian by the addition of a harmonic oscillator potential which is also the generator of special conformal transformations,

$$K = \frac{x^2}{2} \tag{2}$$

The new Hamiltonian is defined by

$$H' = \frac{1}{\omega}(H + \omega^2 K) \tag{3}$$

where we introduced a scale parameter $\omega$ (infrared cutoff). $H'$ has a well-defined vacuum and a discrete spectrum, which can actually be computed exactly,

$$E_n = 2n + 1 + \sqrt{g + 1/4} \tag{4}$$

Notice that the spectrum is independent of the arbitrary scale parameter $\omega$. The supersymmetric case can be dealt with in the same way.

This problem is also encountered in the quantization of a (super)particle moving in the vicinity of the horizon of a black hole [2–5]. The near-horizon geometry of a Reissner-Nordström black hole is $AdS_2 \times S^n$. The isometries of the $AdS_2$ space are conformal symmetries of the particle. As a result, its motion is described by (super)conformal quantum mechanics whose non-relativistic limit is of the form discussed above. The DFF redefinition of the Hamiltonian has a nice interpretation in this case as a redefinition of the time coordinate. The DFF Hamiltonian corresponds to a globally defined time coordinate whereas the conformally invariant definition does not. Thus, the DFF trick appears plausible on physical grounds.

This redefinition has also been applied to more general physical systems, such as the scattering of extended objects [6] and the interaction of extreme black holes [7]. One again obtains a well-defined ground state after the Hamiltonian is redefined following the DFF prescription. However, the physical justification is not as lucid as in the single black hole case.

Here, we present an alternative derivation of the DFF procedure. We show that the redefinition of the Hamiltonian amounts to a different choice of gauge. In the conformally invariant case, we identify an obstruction to the standard gauge-fixing procedure that leads to a modification of the usual quantization rules. This obstruction comes from the boundary of spacetime and is rooted in the fact that the time coordinate is not defined at the boundary. On the other

hand, there is no obstruction in the choice of gauge leading to the DFF Hamiltonian. We conclude that the DFF Hamiltonian corresponds to a good gauge choice, whereas the conformally invariant Hamiltonian does not. Our discussion is based on the standard Faddeev-Popov quantization procedure and is therefore applicable to more general systems, such as the multiple black hole scattering [7], as long as the system has an underlying gauge invariance. Our results indicate that the DFF trick may arise from an obstruction to the Faddeev-Popov procedure at some boundary of moduli space.

Our discussion is organized as follows. In Section II, we apply the Faddeev-Popov procedure to a free particle moving in a fixed background of curved spacetime. We also show how the procedure is equivalent to the commutation rules one obtains from Dirac brackets. In Section III, we extend the procedure by introducing an external electromagnetic field. In Section IV, we specialize to the case of an extreme Reissner-Nordström black hole. We show that the DFF trick is equivalent to a change of gauge. Finally, in Section V, we summarize our conclusions and discuss possible future directions, such as the multi-black-hole moduli space.

## II. NEUTRAL PARTICLE

In this Section, we discuss the quantization of a particle moving in a fixed spacetime background. First, we introduce the path integral for flat spacetime and apply the Faddeev-Popov procedure to fix the gauge. We also show that this is equivalent to the canonical quantization through commutation relations coming from Dirac brackets. We then extend the discussion to general curved spacetime backgrounds.

### A. Flat spacetime

We start our discussion with the more familiar case of a particle of mass $m$ moving freely in flat spacetime. The action is

$$S = \int d\tau \; L \quad , \quad L = \frac{1}{2\eta}\dot{x}^\mu \dot{x}_\mu - \tfrac{1}{2}\eta m^2 \tag{5}$$

The dynamical variables are the spacetime coordinates, $x^\mu$ and we will be working with the signature $(- + + +)$. Varying $\eta$, we obtain the constraint

$$\eta^2 = -\dot{x}^\mu \dot{x}_\mu / m^2 \tag{6}$$

The conjugate momenta are

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{\eta}\,\dot{x}_\mu \quad , \quad P_\eta = 0 \tag{7}$$

The Hamiltonian is

$$H = \dot{x}^\mu P_\mu - L = m\eta\chi \quad , \quad \chi = \frac{1}{2m} P_\mu P^\mu + \tfrac{1}{2} m \tag{8}$$

where we included a mass factor for dimensional reasons. In terms of canonical variables, the action reads

$$S = \int d\tau \, (\dot{x}^\mu P_\mu - m\eta\chi) \tag{9}$$

Therefore, $\eta$ is a Lagrange multiplier enforcing the constraint

$$\chi \equiv \frac{1}{2m} P_\mu P^\mu + \tfrac{1}{2} m = 0 \tag{10}$$

which is the mass-shell condition. This constraint (analogous to Gauss's Law in electrodynamics) generates reparametrizations of $\tau$, through Poisson brackets,

$$\delta x^\mu = \{ \, x^\mu \, , \, \chi \, \}_P \, \delta\tau = \frac{1}{m} \, P^\mu \delta\tau \quad , \quad \delta P_\mu = \{ \, P_\mu \, , \, \chi \, \}_P \, \delta\tau = 0 \tag{11}$$

The orbits of these gauge transformations are straight lines,

$$x^\mu = \frac{P^\mu}{m} \tau + x_0^\mu \tag{12}$$

where $P^\mu, x_0^\mu$ are constant vectors. The family of orbits in the same direction $P^\mu$ fills spacetime. Notice that we can obtain all other families by coordinate transformations (rotations).

To quantize the system, consider the path integral,

$$Z = \mathcal{N} \int \mathcal{D}x \mathcal{D}P \mathcal{D}\eta \; e^{iS} = \mathcal{N} \int \mathcal{D}x \mathcal{D}P \; \delta(\chi) \; e^{i \int d\tau \, \dot{x}^\mu P_\mu} \tag{13}$$

To define it, we need to fix the gauge by imposing the gauge-fixing condition

$$h(x^\mu) = \tau \tag{14}$$

which defines a hyper-surface that cuts each orbit precisely once. Physically, this amounts to choosing $h(x^\mu)$ as the time coordinate. Then its conjugate momentum, $\mathcal{H}$, is the Hamiltonian of the reduced system. Following the standard Faddeev-Popov procedure, we insert

$$1 = \int \mathcal{D}\epsilon \; \{h, \chi\} \; \delta(h - \{h, \chi\}\epsilon - \tau) \tag{15}$$

into the path integral and perform a reparametrization to obtain

$$Z = \mathcal{N} \int \mathcal{D}x \mathcal{D}P \;\; \det\{h, \chi\} \; \delta(h - \tau) \; \delta(\chi) \; e^{i \int d\tau \, \dot{x}^\mu P_\mu} \tag{16}$$

We may integrate over the $\delta$-functions to reduce the dimension of phase space. The reduced system will be described by coordinates $\overline{x}^i$ and conjugate momenta $\overline{P}_i$. The Faddeev-Popov determinant is canceled by the integration over $\delta(\chi)$. The momentum conjugate to $h$ (which is identified with time) plays the rôle of the Hamiltonian $\mathcal{H}$ of the reduced system. The path integral becomes

$$Z = \mathcal{N} \int \mathcal{D}\overline{x}\mathcal{D}\overline{P}\; e^{i\int d\tau\, \dot{\overline{x}}^i \overline{P}_i - \mathcal{H}} \tag{17}$$

Equivalently, we may quantize the system in the operator formalism. To this end, we need to calculate Dirac brackets,

$$\{\, A\,,\; B\,\}_D = \{\, A\,,\; B\,\}_P - \{\, A\,,\; \chi_i\,\}_P\, \{\, \chi_i\,,\; \chi_j\,\}_P^{-1}\, \{\, \chi_j\,,\; B\,\}_P \tag{18}$$

where $i, j = 1, 2$, $\chi_1 = \chi$, $\chi_2 = h$, and promote them to commutators.

As an example, consider the special case $h(x^\mu) = x^0$ (*i.e.,* identify $x^0$ with time). The reduced system is described by the coordinates $\overline{x}^i = x^i$ and the Hamiltonian is

$$\mathcal{H} = -P_0 = \sqrt{P_i P^i + m^2} \tag{19}$$

The commutation relations we obtain from the Dirac brackets are

$$[P_i\,,\; x^j] = -i\delta_i^j \quad , \quad [\mathcal{H}\,,\; x^i] = -i\,\frac{P^i}{\mathcal{H}} \tag{20}$$

which are appropriate for $\mathcal{H}$ given by (19). Having understood the case of flat spacetime, we turn to the problem of a particle moving in a fixed background of curved spacetime.

### B. Curved spacetime

In curved spacetime, one may still use the same action as in flat space (5) and derive the form (9) and hence the constraint (10). The only dfference is that indices are now raised and lowered with the background metric $g_{\mu\nu}$ which is itself a function of the coordinates $x^\mu$. Therefore the gauge transformations (11) get modified to

$$\delta x^\mu = \frac{1}{m}\, P^\mu\, \delta\tau \quad , \quad \delta P_\mu = \frac{1}{m}\, \Gamma_{\nu\lambda\mu} P^\nu P^\lambda\, \delta\tau \tag{21}$$

where $\Gamma_{\nu\lambda\mu}$ are the Christoffel symbols,

$$\Gamma_{\nu\lambda\mu} = \tfrac{1}{2}(\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda}) \tag{22}$$

The orbits are the geodesics, which are solutions to the equations

$$D\,\frac{dx^\mu}{d\tau} \equiv \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\,\frac{dx^\nu}{d\tau}\,\frac{dx^\lambda}{d\tau} = 0 \tag{23}$$

These can also be written in terms of the conjugate momenta,

$$\frac{dP_\mu}{d\tau} + \Gamma_{\nu\lambda\mu} P^\nu P^\lambda = 0 \tag{24}$$

To fix the gauge, we need to find a family of surfaces parametrized by $\tau$ each member of which will intersect the geodesics precisely once. Physically, this implies the choice of a good time coordinate.

## III. CHARGED PARTICLE

If in addition to the curved background the particle interacts with an external electromagnetic field, the action becomes

$$S = \int d\tau \; L \quad , \quad L = \frac{1}{2\eta} \dot{x}^\mu \dot{x}_\mu - \tfrac{1}{2}\eta m^2 + q\dot{x}^\mu A_\mu \tag{25}$$

Varying $\eta$, we obtain the constraint

$$\eta^2 = -\dot{x}^\mu \dot{x}_\mu / m^2 \tag{26}$$

The conjugate momenta are

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{\eta}\, \dot{x}_\mu + qA_\mu \quad , \quad P_\eta = 0 \tag{27}$$

The Hamiltonian is

$$H = \dot{x}^\mu P_\mu - L = m\eta\chi \quad , \quad \chi = \frac{1}{2m}\, \pi_\mu \pi^\mu + \tfrac{1}{2} m \quad , \quad \pi_\mu = P_\mu - qA_\mu \tag{28}$$

In the canonical formalism, the action reads

$$S = \int d\tau \left( \dot{x}^\mu P_\mu - m\eta\chi \right) \tag{29}$$

which is of the same form as in the non-interacting case (9). Therefore, $\eta$ is a Lagrange multiplier enforcing the constraint

$$\chi \equiv \frac{1}{2m}\, \pi_\mu \pi^\mu + \tfrac{1}{2} m = 0 \tag{30}$$

which is the mass-shell condition in the presence of an external vector potential.

The orbits of the gauge transformations ($\tau$ reparametrizations) are the trajectories of the equations of motion (Lorentz force law in curved spacetime)

$$\dot{x}^\mu = \frac{1}{m}\, \pi^\mu \quad , \quad \dot{\pi}_\mu + \frac{1}{m} \Gamma_{\nu\lambda\mu} \pi^\nu \pi^\lambda = \frac{q}{m}\, \pi^\nu F_{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{31}$$

or purely in terms of the coordinates $x^\mu$,

$$\ddot{x}_\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = \frac{q}{m}\, \dot{x}^\nu F_{\mu\nu} \tag{32}$$

The quantization of this system proceeds along the same lines as in the free particle case.

## IV. EXTREME REISSNER-NORDSTRÖM BLACK HOLE

We are now ready to discuss the quantization of a particle moving near an extreme Reissner-Nordström black hole [2–5]. We will discuss both four and five spacetime dimensions. The results in these two cases are similar. The metric in five dimensions is

$$ds^2 = -\frac{1}{\psi^2}dt^2 + \psi d\vec{x}^2 \quad , \quad \psi = 1 + \frac{4Q^2}{\vec{x}^2} \tag{33}$$

and the vector potential is

$$A_0 = \frac{1}{\psi} \quad , \quad \vec{A} = 0 \tag{34}$$

where the vectors live in a four-dimensional Euclidean space. The constant $Q$ is chosen to have the dimension of length and a factor of 4 was including for convenience. Near the horizon, $\psi = 4Q^2/\vec{x}^2$. Using polar coordinates and switching variables to $\psi$, we obtain

$$ds^2 = -\frac{1}{\psi^2}\left(dt^2 - Q^2\, d\psi^2\right) + 4Q^2 d\Omega_3^2 \tag{35}$$

Defining

$$x^{\pm} = t \pm Q\,\psi \tag{36}$$

the metric becomes

$$ds^2 = -\frac{1}{\psi^2}\, dx^+ dx^- + 4Q^2 d\Omega_3^2 \tag{37}$$

and

$$\psi = \frac{x^+ - x^-}{2Q} \tag{38}$$

The vector potential has non-vanishing components

$$A_+ = A_- = \frac{1}{2\psi} \tag{39}$$

In four dimensions, the metric is

$$ds^2 = -\frac{1}{\psi^2}dt^2 + \psi^2 d\vec{x}^2 \quad , \quad \psi = 1 + \frac{Q}{|\vec{x}|} \tag{40}$$

and the vector potential is

$$A_t = \frac{1}{\psi} \quad , \quad \vec{A} = 0 \tag{41}$$

where the vectors live in a three-dimensional Euclidean space. Near the horizon, $\psi = Q/|\vec{x}|$. Using polar coordinates and switching variables to $\psi$, we obtain

$$ds^2 = -\frac{1}{\psi^2}\left(dt^2 - Q^2 d\psi^2\right) + Q^2 d\Omega_2^2 \tag{42}$$

Defining

$$x^\pm = t \pm Q\,\psi \tag{43}$$

the metric becomes

$$ds^2 = -\frac{1}{\psi^2}\, dx^+ dx^- + Q^2 d\Omega_2^2 \tag{44}$$

This is of the same form as in five spacetime dimensions (Eq. (37)), apart from the scale factor in the spherical part of the metric. Thus in both four and five dimensions, spacetime factorizes into a product $AdS_2 \times S^n$ ($n = 2, 3$, respectively). Henceforth, we shall work with $AdS_2$. The only non-vanishing connection coefficients are $\Gamma^\pm_{\pm\pm} = \partial_\pm \ln |g_{+-}|$. Therefore, the geodesic equations for $x^\pm$ are

$$\ddot{x}^\pm \pm (\ln |g_{+-}|)'(\dot{x}^\pm)^2 = \pm\frac{q}{m}\,\dot{x}^\pm F_{+-} \tag{45}$$

where $A = A_+ = A_-$, and $(\ln |g_{+-}|)' = \partial_+ \ln |g_{+-}| = -\partial_- \ln |g_{+-}|$. It is easy to see that the orbits with constant $\psi$ are geodesics,

$$x^+ = x^- + 2\alpha Q \tag{46}$$

where $\psi = \alpha$, parametrized by

$$\tau = \frac{2m\alpha}{q}\,(x^+ + x^-) \tag{47}$$

Motion along these geodesics is generated by the conjugate momentum,

$$H = P_+ + P_- \tag{48}$$

Using $\psi\,\frac{dA}{d\psi} = -A$, $\psi\,\frac{dg_{+-}}{d\psi} = -2g_{+-}$, $F_{+-} = 2\partial_+ A$, it is straightforward to show that the following quantities are gauge-invariant (constant along geodesics)

$$H = P_+ + P_- \quad , \quad D = 2x^+P_+ + 2x^-P_- \quad , \quad K = (x^+)^2 P_+ + (x^-)^2 P_- \tag{49}$$

They obey an $SL(2,\mathbf{R})$ algebra

$$\{\, H \,,\, D \,\} = -2H \quad , \quad \{\, H \,,\, K \,\} = -D \quad , \quad \{\, K \,,\, D \,\} = 2K \tag{50}$$

reflecting the symmetry of the $AdS_2$ spacetime. $H, D$, and $K$ generate time translations, dilatations, and special conformal transformations, respectively. The brackets may be Poisson

or Dirac, so this is also an algebra of the gauge-fixed system, as expected. The constraint (generator of gauge transformations) $\chi \equiv \frac{1}{2m}\pi_\mu\pi^\mu + \frac{1}{2}m = 0$ reads

$$-4\psi^2 P_+ P_- + 2q\psi(P_+ + P_-) + \frac{L^2}{Q^2} - q^2 + m^2 = 0 \tag{51}$$

where $L^2 = \hat{g}^{ij} P_i P_j$ is the square of the angular momentum operator. The simplest gauge-fixing condition to impose is

$$h(x^+, x^-) = x^+ + x^- = \tau \tag{52}$$

In this case, the Hamiltonian is

$$\mathcal{H} = -H = -P_+ - P_- \tag{53}$$

Using the constraint, we obtain

$$\mathcal{H} = \frac{1}{\psi}\left(-q + \sqrt{m^2 + (\psi^2 P_\psi^2 + L^2)/Q^2}\right) \tag{54}$$

The other two operators in the $SL(2, \mathbf{R})$ algebra can be written as

$$D = 2\hat{t}H + 2\psi P_\psi \quad , \quad K = (Q^2\psi^2 + \widehat{t^2})H + 2\hat{t}\psi P_\psi \tag{55}$$

where $\hat{t}$ and $\widehat{t^2}$ are operators satisfying $\dot{\hat{t}} \equiv \{\hat{t}\,,\, H\} = 1$ and $\dot{\widehat{t^2}} \equiv \{\widehat{t^2}\,,\, H\} = 2\hat{t}$, respectively. After some algebra, we find

$$\hat{t} = \psi P_\psi / H \quad , \quad \widehat{t^2} = Q^2\psi^2 - 2qQ^2\psi/H - 2\psi^2 P_\psi^2/H^2 \tag{56}$$

Therefore,

$$D = 4\psi P_\psi \quad , \quad K = 2Q^2\psi^2 H - 2qQ^2\psi \tag{57}$$

In the non-relativistic limit,

$$\mathcal{H} = \frac{\psi P_\psi^2}{2mQ^2} + \frac{g}{8Q^2\psi} \quad , \quad D = 4\psi P_\psi \quad , \quad K = (g/4 - 2qQ^2)\psi \tag{58}$$

where $g = 8Q^2(m-q) + 4L^2/m$. Let us introduce a new variable $x$, defined by

$$\psi = \frac{x^2}{4Q^2} \tag{59}$$

In the limit $q \to m$, $Q \to \infty$, so that $Q^2(m-q)$ remains fixed, we have

$$\mathcal{H} = \frac{P^2}{2m} + \frac{g}{2x^2} \quad , \quad D = 2xP \quad , \quad K = -\tfrac{1}{2}mx^2 \tag{60}$$

where $P$ is the momentum conjugate to $x$. This system does not have a well-defined vacuum. The question then arises whether the underlying theory is inherently sick. One may apply the DFF trick to produce a Hamiltonian system with a well-defined ground state. The DFF trick can be understood in this case as a different choic of time coordinate leading to a different Hamiltonian. From our point of view, any two choices of time coordinates should be equivalent to each other, for they merely correspond to different gauge choices. Since the underlying theory is gauge-invariant, all gauge choices should be equivalent to each other.

Before we discuss the vacuum problem in conjunction with the gauge-fixing procedure, we shall introduce a class of gauges that lead to a Hamiltonian system with a well-defined ground state. Let the gauge-fixing condition be

$$h(x^+, x^-) = \arctan\left(\frac{\omega x^+ + \omega x^-}{1 - \omega^2 x^+ x^-}\right) = \tau \tag{61}$$

where $\omega$ is an arbitrary scale. Differentiating with respect to $\tau$, we obtain

$$\partial_+ h\, \dot{x}^+ + \partial_- h\, \dot{x}^- = 1 \quad , \quad \partial_\pm h = \frac{\omega}{1 + \omega^2 (x^\pm)^2} \tag{62}$$

The Hamiltonian is

$$\mathcal{H} = -\frac{P_+}{\partial_+ h} - \frac{P_-}{\partial_- h} = -\frac{1}{\omega}\left(H + \omega^2 K\right) \tag{63}$$

In the non-relativistic limit,

$$\mathcal{H} = \frac{P^2}{2m\omega} + \frac{g^2}{2\omega x^2} + \tfrac{1}{2} m\omega x^2 \tag{64}$$

which has a well-defined vacuum. All these gauges are of course equivalent. Therefore, no physical quantities should depend on the scale parameter $\omega$. In particular, notice that the spectrum in the non-relativistic limit is independent of $\omega$.

It is puzzling that there exists a gauge (Eq. (52)) in which the vacuum is not well-defined. To resolve the puzzle, let us follow the Faddeev-Popov gauge-fixing procedure a little more carefully. We need to insert (15) into the path integral and then perform an inverse gauge transformation to eliminate the gauge parameter. In doing so, we encounter an obstruction at the boundary of spacetime. Under a gauge transformation, the change in the action is

$$\delta S = \int d\tau\, \frac{d}{d\tau}(\delta x^\mu P_\mu) - \int d\tau\, \epsilon \dot{\chi} \tag{65}$$

Since $\dot{\chi} = \{\chi\,,\, \chi\} = 0$, we conclude that the action changes by a total derivative,

$$\delta S = \int d\tau\, \frac{d}{d\tau}(\delta x^\mu P_\mu) \tag{66}$$

We have

$$\delta x^\mu = \{x^\mu\,,\,\chi\}\,\epsilon = \frac{\partial \chi}{\partial P_\mu}\,\epsilon \tag{67}$$

therefore,

$$\delta S = P_\mu \frac{\partial \chi}{\partial P_\mu}\,\epsilon \,\bigg|_\partial \tag{68}$$

Notice that, if the generator of gauge transformations, $\chi$, is quadratic in the momenta (as in the free particle case (10)), then the boundary contribution vanishes after imposing the constraint $\chi = 0$. In our case, $\chi$ (Eq. (30)) is not quadratic in the momenta, due to the presence of the vector potential. Therefore $\delta S \neq 0$ and we cannot in general get rid of the gauge parameter on the boundary of spacetime. In the set of gauges (61), there is no boundary contribution, because the Faddeev-Popov determinant vanishes there. Indeed,

$$\{\,h\,,\,\chi\,\} \propto \frac{P_+}{1+\omega^2(x^+)^2} + \frac{P_-}{1+\omega^2(x^-)^2} \tag{69}$$

which vanishes as $x^\pm \to \infty$. For the gauge (52), we obtain a boundary contribution to the path integral,

$$\int_\partial d\epsilon d^D x d^D P \,\{\,h\,,\,\chi\,\}\,\delta(h-\tau)\delta(\chi)\,\exp\left\{iP_\mu \frac{\partial \chi}{\partial P_\mu}\epsilon\right\} \tag{70}$$

This obstruction is absent when $\{\,h\,,\,\chi\,\} = 0$ at the boundary. Physically, this condition implies that the boundary of spacetime is invariant under transformations generated by $h$, which is the time coordinate after gauge-fixing ($h = \tau$). In other words, the boundary is fixed under time translations. Thus the time coordinate (52) is not a good global coordinate and leads to an obstruction in the gauge invariance of the theory. Integrating over the gauge parameter, we obtain an additional constraint at the boundary,

$$P_\mu \frac{\partial \chi}{\partial P_\mu}\bigg|_\partial = 0 \tag{71}$$

This alters the standard commutation relations and the eigenvalue problem for the Hamiltonian (54) or (60). We have not carried out an explicit computation. This would involve the introduction of a regulator which would break gauge invariance. Nevertheless, the resulting system should be equivalent to the one obtained through the other choices of gauge due to the gauge invariance of the theory.

To summarize, the naïve identification of the time coordinate (52) leads to an obstruction in the gauge-fixing procedure for the path integral. If this obstruction is accounted for by an appropriate modification of the commutation relations, this choice of the time coordinate leads to a well-defined Hamiltonian problem. The Hamiltonian system thus obtained is equivalent to applying the DFF trick [1], or identifying the time coordinate as in Eq. (61) [5]. The latter is merely a different gauge choice in a gauge-invariant theory.

## V. CONCLUSIONS

We considered the problem of quantization of a charged particle in the vicinity of the horizon of an extreme Reissner-Nordström black hole. In this case, the vacuum is not well-defined unless a special choice of the time coordinate is made [5]. This has been shown to be equivalent to the DFF trick for conformal quantum mechanics [1]. We approached this problem through the path integral and the Faddeev-Popov gauge-fixing procedure. We showed that the DFF trick can be understood in terms of the standard Faddeev-Popov procedure.

We started with a general discussion of the quantization of a particle in the presence of a background metric field as well as an external vector potential. We performed the standard Faddeev-Popov procedure in the canonical formalism and showed its connection to commutation relations through Dirac brackets. We then applied the procedure to the case of interest (extreme Reissner-Nordström black hole). We showed that the naïve identification of time coordinate (which leads to a Hamiltonian system with no well-defined ground state) corresponds to a gauge which is not "good." We found that in this gauge the Faddeev-Popov procedure encounters an obstruction at the boundary of spacetime introducing an additional constraint there. This alters the standard commutation relations and the eigenvalue problem for the attendant Hamiltonian system. We did not calculate the effects of this obstruction explicitly. This would require the introduction of a regulator which would break gauge invariance explicitly and therefore alter the commutation rules. Instead, we exhibited another set of gauges where no obstruction existed on the boundary. We showed that this set of gauges led to a Hamiltonian system with a well-defined vacuum, equivalent to the one obtained through the DFF trick [1].

A choice of gauge is equivalent to a choice of time coordinate, because gauge transformations are reparametrizations of the proper time of the particle. The above discussion shows that all choices of the time coordinate are equivalent since they correspond to different gauge choices in a gauge-invariant theory. It would be interesting to apply our procedure to multiple black hole scattering, where the study of moduli space seems to necessitate the introduction of the DFF trick [7]. It is hard to see how this can be justified in terms of a "proper" choice of a time coordinate (which time coordinate?). However, the underlying theory is a gauge theory, so the Faddeev-Popov procedure should be applicable. Our results indicate that one may encounter an obstruction to the implementation of the procedure, perhaps on some boundary of moduli space, which would lead to an alteration of the naïve Hamiltonian system similar to the DFF trick. Work in this direction is in progress.

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